On the number of L-shapes in embedding dimension four numerical semigroups

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Abstract

Minimum distance diagrams, also known as *L–shapes*, have been used to study some properties related to *weighted Cayley digraphs* of *degree* two and *embedding dimension three numerical semigroups*. In this particular case, it has been shown that these discrete structures have at most two related L–shapes. These diagrams are proved to be a good tool for studying *factorizations* and the *catenary degree* for semigroups and *diameter* and *distance* between vertices for digraphs.

This maximum number of L–shapes has not been proved to be kept when increasing the degree of digraphs or the embedding dimension of semigroups. In this work we give a family of embedding dimension four numerical semigroups S_n, for odd $n \geq 5$, such that the number of related L–shapes is $\frac{n+3}{2}$. This family has her analog to weighted Cayley digraphs of degree three.

Therefore, the number of L–shapes related to numerical semigroups can be as large as wanted when the embedding dimension is at least four. The same is true for weighted Cayley digraphs of degree at least three. This fact has several implications on the combinatorics of factorizations for numerical semigroups and minimum paths between vertices for weighted digraphs.

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1. Introduction

Minimum Distance Diagrams (MDD for short) have been used in different discrete structures to study several optimization problems. Most known examples of this use are metrical optimization problems in Cayley digraphs on cyclic finite Abelian groups and several questions in numerical semigroups. Frobenius number computation, factorization related properties and the study of Apéry sets are some applications in the latter example.

A *Cayley digraph* $G = C(N; s_1, \dots, s_k; p_1, \dots, p_k)$ on the cyclic finite Abelian group $\mathbb{Z}_N$ generated by the generator set $B = \{s_1, \dots, s_k\} \subset \mathbb{Z}_N \setminus \{0\}$, $\gcd(N, s_1, \dots, s_k) = 1$, is a directed graph with set of vertices $V = \mathbb{Z}_N$ and

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set of arcs $A = \left\{ m \xrightarrow{p_i} (m + s_i) \pmod N \mid m \in V, 1 \le i \le k \right\}$, where p_i is the weight of the arc defined by s_i, $i \in \{1, \ldots, k\}$. The *length* of a path in G is the sum of the weights of the arcs in the path. A *minimum path* from m to n is a connecting path from m to n with minimum length in G. The *distance* from m to n, $\mathrm{d}(m,n)$, is the length of a minimum path from m to n. The *diameter* of G, $\mathrm{D}(G)$, is the maximum of the distances between pairs of vertices in G. The metric on G depends on the weights of his arcs.

Let us consider unit cubes in $\mathbb{R}^k$. Each unit cube $[i_1, i_1+1] \times [i_2, i_2+1] \times \cdots \times [i_k, i_k+1] \in \mathbb{R}^k$ has integral coordinates $(i_1, \ldots, i_k) \in \mathbb{Z}^k$ and it is usually labelled with the vertex $i_1 s_1 + \cdots + i_k s_k \pmod N$ (sometimes it is also labelled with his 'weight' $i_1 p_1 + \cdots + i_k p_k$). We denote the unit cube with coordinates $(i_1, \ldots, i_k)$ by $[\![i_1, \ldots, i_k]\!]$. Let $\le$ be the usual partial ordering in $\mathbb{N}^k$. A unified definition of minimum distance diagrams was given by P. Sabariego and F. Santos in 2009 [9, Definition 2.1] although other authors used this concept, see for instance Fiol et al. [5] and Rödseth [7]. Following the definition of [9], a *minimum distance diagram* $\mathcal{H}$ related to G is a connected set of N unit cubes in $\mathbb{R}^k$ with different vertex label and the following two properties

(1) if $u = [\![i_1, \ldots, i_k]\!] \in \mathcal{H}$, then the weight $\|u\| = i_1 p_1 + \cdots + i_k p_k$ is minimum over all cubes with coordinates $(j_1, \ldots, j_k) \in \mathbb{N}^k$ fulfilling $j_1 s_1 + \cdots + j_k s_k \equiv i_1 s_1 + \cdots + i_k s_k \pmod N$,

(2) if $v = [\![j_1, \ldots, j_k]\!]$ is a cube with $(j_1, \ldots, j_k) \le (i_1, \ldots, i_k)$, then $v \in \mathcal{H}$.

It have been proved these diagrams are L-shaped regions of the plane (or rectangles) when $k = 2$ ([5]). For this reason they are called *L-shapes* when $k = 2$ and *hyper L-shapes* when $k \ge 3$.

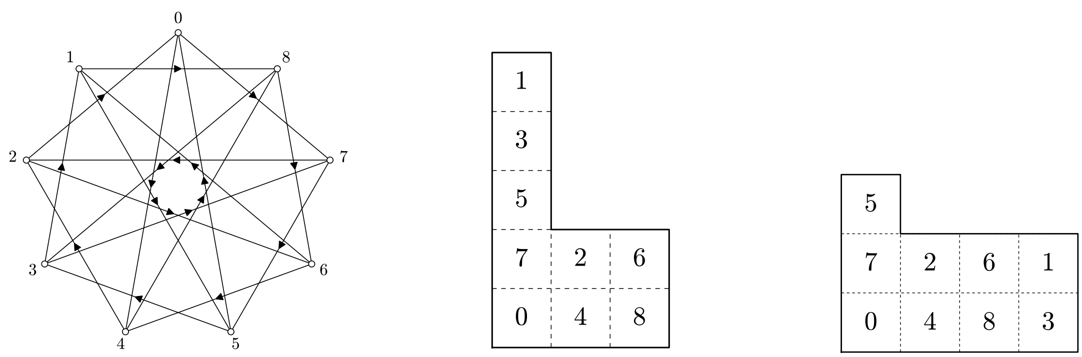


Figure 1: $C(\mathbb{Z}_9; 4, 7; 1, 1)$ and two related MDD

Usually, problems in Graph Theory are stated in the non–weighted version of arcs, that is, $p_1 = \cdots = p_k = 1$. For instance, Figure 1 shows two minimum distance diagrams associated with $C(\mathbb{Z}_9; 4, 7; 1, 1)$. There is a significative difference between $k = 2$ and $k \ge 3$. When $k = 2$, it has been shown that these digraphs have two related MDD at most. For $k = 3$, Sabariego and Santos [9] gave an infinite family of digraphs with many associated MDDs. More precisely, given $t \not\equiv 0 \pmod 3$, set $m = 2 + t + t^2$; then, the digraph $G_t = C(m(m-1); 1+m, 1+mt, 1+mt^2; 1, 1, 1)$ has $3(t+2)$ associated MDDs. Taking $t = 2$, the digraph $C(\mathbb{Z}_{56}; 9, 17, 33; 1, 1, 1)$ has 12 different associated MDDs that have been depicted in Figure 2.

Changing weights $\{p_1, \ldots, p_k\}$ provides a different metric on the digraph. Thus, the number of minimum distance diagrams eventually decreases. Non-weighted version of the digraph has more related MDDs than the weighted one. For instance, changing $p_1 = p_2 = 1$ to $p_1 = 2$ and $p_2 = 3$, only the second diagram in Figure 1 is an MDD related to $C(\mathbb{Z}_9; 4, 7; 2, 3)$. Also taking $p_1 = 9$, $p_2 = 17$ and $p_3 = 33$ only the fifth and the last diagrams in Figure 2 are MDDs related to $G_2 = C(\mathbb{Z}_{56}; 9, 17, 33; 9, 17, 33)$. Table 1 shows the number of MDDs resulting from taking the weights $p_1 = 1 + m$, $p_2 = 1 + mt$ and $p_3 = 1 + mt^2$ in G_t. Important types of weights are $p_i = s_i$ for all i. This choice of weights can modelize some properties of numerical semigroups on digraphs.

Now, we look at numerical semigroups to translate the Minimum Distance Diagrams to this setting.

Given $n_1, \ldots, n_k \in \mathbb{N}$ with $\gcd(n_1, \ldots, n_k) = 1$, the *numerical semigroup* generated by $A = \{n_1, \ldots, n_k\}$ is the set $S = \langle n_1, \ldots, n_k \rangle = \{x_1 n_1 + \cdots + x_k n_k : (x_1, \ldots, x_k) \in \mathbb{N}^k\}$. We use the notation $Ax = x_1 n_1 + \cdots + x_k n_k$ for $x \in \mathbb{N}^k$.

t	2	4	5	7	8	10	11	13	14	16	17	19	20
N_t	56	462	992	3306	5402	12432	17822	33672	44732	74802	94556	145542	177662
# non–w.	12	18	21	27	30	36	39	45	48	54	57	63	66
# w.	2	2	2	2	2	2	12	2	2	14	2	2	2

Table 1: Number of related MDDs for weighted and non–weighted G_t, $t \not\equiv 0 \pmod 3$

Let S be a numerical semigroup minimally generated by A (no proper subset of A generates the same semigroup), and let k be the cardinality of A, which is known as the *embedding dimension* of S.

The *set of factorizations* of $s \in S$ is $\mathsf{Z}(s) = \{x \in \mathbb{N}^k \mid Ax = s\}$. For a subset X of S, let $\mathsf{Z}(X) = \bigcup_{x \in X} \mathsf{Z}(x)$ (this union is disjoint).

For $a, b \in \mathbb{Z}$, we write $a \leq_S b$ if $b - a \in S$. Let $C \subseteq S$. We say that C is *closed* if whenever $a \in C$ and $b \leq_S a$, then $b \in C$.

Definition 1. *Fixed C a nonempty closed subset of S, we say that $L \subset \mathbb{N}^k$ is an* L-shape *associated to C if the following two properties hold*

(C1) the map $x \mapsto Ax$ is a bijection from L to C ($\#(\mathsf{Z}(c) \cap L) = 1$ for all $c \in C$),

(C2) if $x \in L$, then $y \in L$ for every $y \in \mathbb{N}^k$ with $y \leq x$.

A particular case of closed sets in a numerical semigroup are the Apéry sets. Let $m \in S \setminus \{0\}$. The *Apéry set* of m in S is the set

$$\mathrm{Ap}(S, m) = \{s \in S \mid s - m \notin S\}.$$

It can be easily shown that $\mathrm{Ap}(S, m) = \{w_0, \ldots, w_{m-1}\}$, where $w_i = \min\{s \in S \mid s \equiv i \pmod m\}$ for $i \in \{0, \ldots, m-1\}$. In particular, the cardinality of $\mathrm{Ap}(S, m)$ is m (see for instance [8, Lemma 2.4]).

Given a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle$, let us consider the related digraph

$$G_S = C(n_k; n_1, \ldots, n_{k-1}; n_1, \ldots, n_{k-1}).$$

Many metric properties of the digraph G_S give information on the semigroup S. For instance, for $k = 3$, generic properties of the sets of factorizations are studied in [1].

Let us denote the weight of the unit cube $u = [\![i_1, \ldots, i_{k-1}]\!]$ as $\|u\| = n_1 i_1 + \cdots + n_{k-1} i_{k-1}$. If $\mathcal{H}$ is a minimum distance diagram associated with G_S, then it can be seen that

$$\mathcal{W}_{\mathcal{H}} = \{\|u\| \mid u \in \mathcal{H}\} = \mathrm{Ap}(S, n_k).$$

Thus, an L-shape related to S is equivalent to a minimum distance diagram associated with G_S. It is also known that for embedding dimension three, S admits at most two L-shapes. So, there is a natural question arising from this equivalence for embedding dimension larger than three: are there numerical semigroups with a number of related L-shapes as large as we want? As far as we know, there is no related work in the bibliography.

2. A distinguished infinite family of 4–semigroups

We have implemented the construction of L-shapes in the `numericalsgps` ([4]) `GAP` ([6]) package. Computer evidence convinced us to look for a parameterized family of embedding dimension four numerical semigroups with as many L-shapes as desired.

Bresinsky in [3] gave a family of numerical semigroups with embedding dimension four with arbitrary large minimal presentations (embedding dimension three numerical semigroups admit minimal presentations with at most three elements; thus the analogy with our setting). Unfortunately all the elements in his family have exactly two L-shapes.

In order to look for primitive elements, we developed an algorithm in [2] that gave us some light to find the family of semigroups that we present in this paper.

Let n be an odd integer greater than or equal to five. Set

$$T = \langle n, 3n-2, 3n-1\rangle \text{ and } S = T \cup \{\mathrm{F}(T)\} = \langle n, 3n-2, 3n-1, \mathrm{F}(T)\rangle,$$

where $\mathrm{F}(T)$ denotes $\max(\mathbb{Z}\setminus T)$, the Frobenius number of T.

Observe that a minimal generating set for T is $\{n, 3n-2, 3n-1\}$. Hence there exists an epimorphism $\phi : \mathbb{N}^3 \to T$, $\phi(a,b,c) = an + b(3n-2) + c(3n-1)$. The kernel of ϕ, $\ker\phi = \{(\alpha,\beta) \in \mathbb{N}^3\times\mathbb{N}^3 \mid \phi(\alpha) = \phi(\beta)\}$ is a congruence. A minimal generating system of $\ker\phi$ as a congruence is known as a *minimal presentation* for T. Minimal presentations turn out to be a key tool in the study of factorizations.

In order to find a minimal presentation of T we must find the least multiple of each generator that belongs to the semigroup spanned by the other two (see for instance [8, Example 8.23]).

Clearly, $2(3n-1) = 3n + (3n-2)$. Observe that if we look for the least multiple of $3n-2$ that belongs to $\langle n, 3n-1\rangle$, we have to solve the equation $a(3n-2) = bn + c(3n-1)$. Thus we are looking for $a, b, c \in \mathbb{N}$, $a \neq 0$ such that $(3a-b-3c)n = 2a-c$. Hence we must solve

$$\begin{cases} 3a - b - 3c = k, \\ 2a - c = kn, \end{cases}$$

with $k \in \mathbb{N}$. We get the parametrized solutions $a = \frac{kn+c}{2}$ and $b = \frac{k(3n-2)-3c}{2}$. For $k = 0$ there is no nonnegative solution to the equations, and thus the least possible a is reached for $k = 1$, and since n is odd, in order to get $a \in \mathbb{N}$, c cannot be zero. Hence the least possible value of a is $\frac{n+1}{2}$, and then $b = \frac{3n-5}{2} \in \mathbb{N}$ and $c = 1$. So we already have two relations among the generators:

$$\begin{aligned} 2(3n-1) &= 3n + 1(3n-2), \\ \tfrac{n+1}{2}(3n-2) &= \tfrac{3n-5}{2}n + 1(3n-1). \end{aligned}$$

In light of [8, Lemma 10.19], the third relation can be obtained from these two by "adding" them together:

$$\frac{3n+1}{2}n = \frac{n-1}{2}(3n-2) + 1(3n-1).$$

Therefore, a minimal presentation for T is

$$\left\{((0,0,2),(3,1,0)), \left(\left(0,\frac{n+1}{2},0\right),\left(\frac{3n-5}{2},0,1\right)\right), \left(\left(\frac{3n+1}{2},0,0\right),\left(0,\frac{n-1}{2},1\right)\right)\right\}. \tag{1}$$

From [8, Proposition 2.20 and Lemma 10.20], we obtain that the set of pseudo-Frobenius numbers of S, $\mathrm{PF}(T) = \{z \in \mathbb{Z} \mid z + T\setminus\{0\} \subseteq T\}$, is

$$\mathrm{PF}(T) = \left\{\frac{3n-7}{2}n + 1, \frac{3n-7}{2}n + 2\right\}.$$

In particular $\mathrm{F}(T) = \frac{3n-7}{2}n + 2$ and $\mathrm{F}(T) - 1 \notin T$. This implies that $\mathrm{F}(S) = \mathrm{F}(T) - 1$ and as a consequence of this, $\max \mathrm{Ap}(S, \mathrm{F}(T)) = 2\,\mathrm{F}(T) - 1 = n(3n-7) + 3$ ([8, Proposition 2.12]).

2.1. Factorizations of the elements of the Apéry set

In this section we describe what are the factorizations of the elements of $\mathrm{Ap}(S,\mathrm{F}(T))$. We are going to use extensively the fact that $\mathrm{Ap}(S,\mathrm{F}(T))$ is a closed set, as it has been remarked before. Also every element s in $\mathrm{Ap}(S,\mathrm{F}(T))$ is in T, and thus we identify the set $\mathsf{Z}(s)$ with a subset of $\mathbb{N}^3$; indeed $\mathsf{Z}(s)$ is in one-to-one correspondence with $\phi^{-1}(s)$.

Lemma 2. *Let $s \in \mathrm{Ap}(S,\mathrm{F}(T))$. There exists exactly one factorization $(x,y,z) \in \mathsf{Z}(s)$ such that $z < 2$, $y < \frac{n+1}{2}$ and $x < \frac{3n-1}{2}$.*

Proof. We can use the minimal presentation of T to obtain one factorization of s with the third coordinate less than two and the second less than $\frac{n+1}{2}$. Notice that as $n\frac{3n+1}{2} - \mathrm{F}(T) = 4n-2 \in S$, we have $n\frac{3n+1}{2} \notin \mathrm{Ap}(S,\mathrm{F}(T))$, and so the third relation is never used on the factorizations of s. As $n\frac{3n-1}{2} - \mathrm{F}(T) = 3n-2 \in S$, the first coordinate must be less than $\frac{3n-1}{2}$.

Now assume that there is another $(x', y', z') \in \mathsf{Z}(s)$ with $z' < 2$, $x' < \frac{3n-1}{2}$ and $y < \frac{n+1}{2}$. From the definition of minimal presentation there should be a chain of reductions going from (x, y, z) to (x', y', z') by using the relations in the minimal presentation. We already know that the third relation cannot be used. Also as $z, z' < 2$ and $y, y' < \frac{n+1}{2}$, the only possibility is that either $(3, 1, 0) < (x, y, z)$ or $\left(\frac{3n-5}{2}, 0, 1\right) < (x, y, z)$. If $\left(\frac{3n-5}{2}, 0, 1\right) < (x, y, z)$, then $z = 1$ and $\left(x - \frac{3n-5}{2}, y + \frac{n+1}{2}, 0\right) \in \mathsf{Z}(s)$. Also $x < \frac{3n-1}{2}$, whence $x - \frac{3n-5}{2} < \frac{3n-1}{2} - \frac{3n-5}{2} = 2$. So to meet this new factorization we cannot apply the first relation, which means that we can only, eventually, use the second one obtaining always factorizations with second coordinate greater than $\frac{n+1}{2}$. Assume now that $(3, 1, 0) < (x, y, z)$. Then $(x - 3, y - 1, z + 2) \in \mathsf{Z}(s)$. Again, as $x < \frac{3n-1}{2}$, $x - 3 < \frac{3n-5}{2}$, and $y - 1 < \frac{n+1}{2}$. So we cannot apply here the second relation. This means that we could only apply here the first one, obtaining in any case factorizations with last coordinate greater than two. $\square$

We can define an injective mapping from $\mathrm{Ap}(S, \mathrm{F}(T))$ to $\mathbb{N}^3$ that assigns to every $s \in \mathrm{Ap}(S, \mathrm{F}(T))$ the only factorization (x, y, z) fulfilling the conditions of Lemma 2. Let us denote this map by

$$\mathrm{nf} : \mathrm{Ap}(S, \mathrm{F}(T)) \to \mathbb{N}^3.$$

For $s \in \mathrm{Ap}(S, \mathrm{F}(T))$, we will say that $\mathrm{nf}(s)$ is the *normal form* of s. As usual, given $X \subseteq \mathrm{Ap}(S, \mathrm{F}(T))$, we write $\mathrm{nf}(X) = \{\mathrm{nf}(s) \mid s \in X\}$.

Lemma 3. *Under the standing hypothesis,*

$$\left\{\left(\frac{3n-3}{2}\right)n + \left(\frac{n-3}{2}\right)(3n-2), \quad n + \left(\frac{n-1}{2}\right)(3n-2), \right.$$
$$\left(\frac{3n-5}{2}\right)n + \left(\frac{n-5}{2}\right)(3n-2) + (3n-1),$$
$$\left.\left(\frac{3n-3}{2}\right)n + \left(\frac{n-7}{2}\right)(3n-2) + (3n-1)\right\} \subseteq \mathrm{Ap}(S, \mathrm{F}(T)).$$

Proof. Clearly, $\left(\frac{3n-3}{2}\right)n + \left(\frac{n-3}{2}\right)(3n-2) \in S$, and $\left(\frac{3n-3}{2}\right)n + \left(\frac{n-3}{2}\right)(3n-2) - \mathrm{F}(T) = \mathrm{F}(T) - 1 \notin S$.

As $n + \left(\frac{n-1}{2}\right)(3n-2) - \mathrm{F}(T) = 2n - 1 \notin S$, it easily follows that $n + \left(\frac{n-1}{2}\right)(3n-2)$ is in the Apéry set.

From identity $\left(\frac{3n-5}{2}\right)n + \left(\frac{n-5}{2}\right)(3n-2) + (3n-1) - \mathrm{F}(T) = \mathrm{F}(T) - n$, it follows the third element also belongs to this Apéry set.

Finally, the last element in the list is in the Apéry since $\left(\frac{3n-3}{2}\right)n + \left(\frac{n-7}{2}\right)(3n-2) + (3n-1) - \mathrm{F}(T) = \mathrm{F}(T) - (3n-2) \notin S$.

$\square$

Let

$$F = \left\{(x, y, 0) \in \mathbb{N}^3 \mid x \le \frac{3n-3}{2}, y \le \frac{n-3}{2}\right\} \cup \left\{(x, y, 0) \in \mathbb{N}^3 \mid x \le 1, y = \frac{n-1}{2}\right\}$$
$$\cup \left\{(x, y, 1) \in \mathbb{N}^3 \mid x \le \frac{3n-3}{2}, y \le \frac{n-7}{2}\right\} \cup \left\{(x, y, 1) \in \mathbb{N}^3 \mid x \le \frac{3n-5}{2}, y = \frac{n-5}{2}\right\}.$$

We will denote respectively F_1, F_2, F_3 and F_4 the four sets in the above representation of F.

Lemma 4. *The map* $\mathrm{nf} : \mathrm{Ap}(S, \mathrm{F}(T)) \to F$ *is a bijection.*

Proof. By using Lemma 3 and the fact that $\mathrm{Ap}(S, \mathrm{F}(T))$ is closed, we have $xn + y(3n-2) \in \mathrm{Ap}(S, \mathrm{F}(T))$ for all $(x, y, 0) \in F_1 \cup F_2$. Also, the elements $xn + y(3n-2) + (3n-1)$ belong to the Apéry $\mathrm{Ap}(S, \mathrm{F}(T))$, with $(x, y, 1) \in F_3 \cup F_4$.

All the elements we have obtained so far are different by Lemma 2. Counting them all, we get

$$\left(\frac{3n-1}{2}\right)\left(\frac{n-1}{2}\right) + 2 + \left(\frac{3n-1}{2}\right)\left(\frac{n-5}{2}\right) + \frac{3n-3}{2} = \frac{3n^2-7n}{2} + 2 = \mathrm{F}(T).$$

And as the cardinality of $\mathrm{Ap}(S, \mathrm{F}(T))$ is precisely $\mathrm{F}(T)$, we obtain the desired result. $\square$

Observe that F is an L-shape associated with $\mathrm{Ap}(S, \mathrm{F}(T))$.

2.2. Computing the number of factorizations of elements in $\mathrm{Ap}(S, \mathrm{F}(T))$

Lemma 5. *An element* $s \in \mathrm{Ap}(S, \mathrm{F}(T))$ *has only one factorization if and only if* $\mathrm{nf}(s)$ *satisfies one of the following disjoint conditions:*

1) $\mathrm{nf}(s) = (x, 0, 0)$ *with* $0 \le x \le \frac{3n-3}{2}$,

2) $\mathrm{nf}(s) = (x, 0, 1)$ *with* $0 \le x \le \frac{3n-7}{2}$,

3) $\mathrm{nf}(s) = (2, y, 0)$ *with* $1 \le y \le \frac{n-3}{2}$,

4) $\mathrm{nf}(s) = (x, y, 0)$ *with* $0 \le x \le 1$ *and* $1 \le y \le \frac{n-1}{2}$,

5) $\mathrm{nf}(s) = (x, y, 1)$ *with* $0 \le x \le 2$ *and* $1 \le y \le \frac{n-5}{2}$.

Proof. Notice that we are looking for elements in F such that they are not bigger than or equal to (with respect to the usual partial ordering in $\mathbb{N}^3$) any of the six factorizations involved in the minimal presentation of S, (1). So the sufficiency is clear.

As $((0, 0, 2), (3, 1, 0))$ is in the minimal presentation, we have that $z < 2$. We begin with the case $y = 0$.

- If $z = 0$, as we have elements with only one factorization, the factorization $(x, 0, 0)$ must not be bigger than or equal to $(\frac{3n+1}{2}, 0, 0)$. If this were not the case, we could apply the third element of the minimal presentation. Then we are choosing elements in F_1, so only can take $x \le \frac{3n-3}{2}$.
- If $z = 1$, the factorization $(x, 0, 1)$ must be located below $(\frac{3n-5}{2}, 0, 1)$ since, otherwise, $\left(x - \frac{3n-5}{2}, \frac{n+1}{2}, 0\right)$ would be another factorization of $\mathrm{nf}(s)$ (in this case we are lying in F_3, but in this case this is not relevant).

Now we look at the case $y > 0$. In this setting we have $x < 3$. We distinguish either $z = 0$ or $z = 1$. For $z = 0$ we get two subcases.

- If $x = 2$, these elements are in F_1, and consequently we have $y \le \frac{n-3}{2}$.
- If $x \le 1$, we are choosing elements in F_2, so we have $y \le \frac{n-1}{2}$.

Finally, if $y \ge 1$ and $z = 1$, we are taking elements in $F_3 \cup F_4$, whence $y \le \frac{n-5}{2}$. □

Corollary 6. *There exist* $6n - 13$ *elements in* $\mathrm{Ap}(S, \mathrm{F}(T))$ *with only one factorization.*

Proof. We only need to sum the elements in each of the five items from Lemma 5:

$$\frac{3n-1}{2} + \frac{3n-5}{2} + \frac{n-3}{2} + 2\frac{n-1}{2} + 3\frac{n-5}{2} = 6n - 13.$$ □

Define M_i as the set of elements in $\mathrm{Ap}(S, \mathrm{F}(T))$ having i factorizations, i.e.

$$M_i = \{s \in \mathrm{Ap}(S, \mathrm{F}(T)) \mid \ \#\mathsf{Z}(s) = i\}.$$

Lemma 7. *Let* i *be a positive integer such that* $2 \le i \le \frac{n-3}{2}$. *An element* $s \in M_i$ *if and only if* $\mathrm{nf}(s) = (x, y, z)$ *satisfies one of the following disjoint conditions:*

1) $y = i - 2$, $z = 1$ *and* $\frac{3n-5}{2} \le x \le \frac{3n-3}{2}$,

2) $y = i - 1$, $z = 0$ *and* $3(i - 1) \le x \le \frac{3n-3}{2}$,

3) $y = i - 1$, $z = 1$ *and* $3(i - 1) \le x \le \frac{3n-7}{2}$,

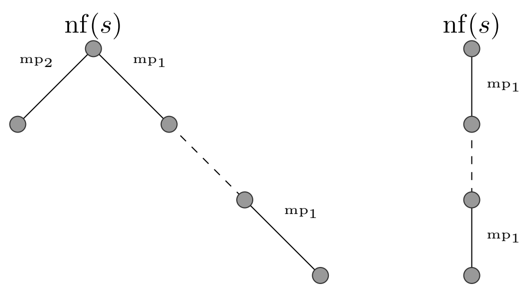
4) $i \le y \le \frac{n-3}{2}$, $z = 0$ *and* $3(i - 1) \le x \le 3i - 1$,

5) $i \le y \le \frac{n-5}{2}$, $z = 1$ *and* $3(i - 1) \le x \le 3i - 1$.

Proof. To construct the set of all factorizations of an element $s \in \mathrm{Ap}(S, \mathrm{F}(T))$, we can start with $\mathrm{nf}(s) = (x, y, z)$, and then, by using the elements of the minimal presentation, as times as possible, we can find the remaining factorizations for s (this is a consequence of [8, Lemma 8.4]). As we will take elements in $\mathrm{Ap}(S, \mathrm{F}(T))$, then, as we have seen in the proof of Lemma 2, only $((0, 0, 2), (3, 1, 0))$ and $\left(\left(0, \frac{n+1}{2}, 0\right), \left(\frac{3n-5}{2}, 0, 1\right)\right)$ can be used in this construction. We denote them as mp_1 and mp_2, respectively. We will refer to operation mp_i when we obtain a new factorization of an element by using the relation mp_i. For instance, from the factorization $(0,0,2)$, by using mp_1, we obtain $(3,1,0)$.

Starting with $(x,y,z) = \mathrm{nf}(s)$, the operation mp_1 only can be applied by subtracting $(3,1,0)$ and adding $(0,0,2)$, since the elements in F have $z \le 1$. Analogously, operation mp_2 only can be applied by subtracting $\left(\frac{3n-5}{2},0,1\right)$ and adding $\left(0,\frac{n+1}{2},0\right)$; we can not subtract $\left(0,\frac{n+1}{2},0\right)$ because the second coordinate of the elements in F are less than or equal to $\frac{n-1}{2}$. Also, the operation mp_2 can only be applied to elements on F with the first coordinate $x = \frac{3n-3}{2}$ or $x = \frac{3n-5}{2}$ and the third coordinate $z = 1$.

We are going to see that in the tree of factorizations obtained by applying mp_1 and mp_2, whenever we apply mp_2, we encounter a leaf, that is, we cannot obtain new factorizations following this path. Actually the tree (rooted in $\mathrm{nf}(s)$) would have the following two possible shapes.



First main idea: we only can apply mp_2 once, and after applying mp_2 we can not use mp_1 anymore (corresponds with the figure on the left in the above picture). When we apply mp_2 to one of the elements in F, we obtain $(x',y',0)$ with $x' \le 1$. So we can not apply mp_2 again because the first coordinate is now either $x' = 0$ or $x' = 1$, which is less than $\frac{3n-5}{2}$. Then, we can not apply operation mp_1 anymore because $x' \le 1$ and $z' = 0$. So we can apply mp_2 no more than one time and only for the elements $s \in \mathrm{Ap}(S,\mathrm{F}(T))$ that have the first coordinate of $\mathrm{nf}(s)$ either $x = \frac{3n-3}{2}$ or $x = \frac{3n-5}{2}$.

Second main idea: after applying mp_1 we cannot use mp_2 (associated to the figure on the right in the above picture). If the first coordinate of $\mathrm{nf}(s)$ is different from $x = \frac{3n-3}{2}$ or $x = \frac{3n-5}{2}$, then we can only apply mp_1 to obtain a new factorization. Afterwards we can not apply mp_2 anymore to any new factorization obtained applying operation mp_1. This is because each time mp_1 is applied, the first coordinate of $\mathrm{nf}(s)$ decreases.

So, in order to count all possible factorizations obtained from $\mathrm{nf}(s)$ with $s \in \mathrm{Ap}(S,\mathrm{F}(T))$, we must enumerate how many times we can apply operation mp_1 first. Then, whenever the first coordinate is $x \ge \frac{3n-5}{2}$, we can construct an extra factorization with the use of mp_2.

Thus, for every element in F, we try to apply operation mp_2 to $\mathrm{nf}(s)$ and then, we try to apply operation mp_1 to $\mathrm{nf}(s)$ as many times as possible.

To justify the five cases in the statement, we show first that when $y < i-2$ there are not i factorizations. So take $(x,y,z) \in F$ with $y < i-2$. Recall that we can apply mp_2 once, at most. When applying operation mp_1, we subtract one on the second coordinate. So in this settings, we can apply at most $i-3$ times mp_1 and, eventually, one more time mp_2. This process adjoins up to $i-3+1 = i-2$ new factorizations, that added to the original (x,y,z), cannot give i factorizations.

Now we consider $s \in M_i$ with $\mathrm{nf}(s) = (x,y,z)$.

- If the second coordinate is $y = i-2$, as we have seen above, we can obtain $i-2$ new factorizations with mp_1. So to get exactly i factorizations, we need an extra one by applying mp_2. But this is possible if and only if $z = 1$ and $x \in \left\{\frac{3n-3}{2},\frac{3n-5}{2}\right\}$. In these cases, as $\frac{n-3}{2} \ge i$, we get $x \ge \frac{3n-5}{2} > \frac{3n-9}{2} \ge 3i$, and the first coordinate is large enough to subtract $(i-2)$ times 3. This corresponds with *1)*.
- Now we assume $y = i-1$. We can construct $i-1$ new factorizations by applying the mp_1 operation and, adding the original one, we obtain i factorizations. So, it is necessary that operation mp_2 can not be applied. Hence, we have $z = 0$ and $x \le \frac{3n-3}{2}$ (when $(x,y,z) \in F_1$), or $z = 1$ and $x \le \frac{3n-7}{2}$ (when $(x,y,z) \in F_3$) as in this case, if $x \ge \frac{3n-5}{2}$ we can apply mp_2. In both cases, we also need that $x \ge 3(i-1)$. This yields *2)* and *3)*.

- If we take $y \geq i$, we need to have $x \leq 3i-1$ to ensure that we only can apply mp_1 $i-1$ times. As $x \leq 3i-1$ and $i \leq \frac{n-3}{2}$, we have $x \leq 3i-1 \leq \frac{3n-11}{2}$, so operation mp_2 can not be applied. In these cases, we will need again the extra condition $x \geq 3(i-1)$. Finally, recall that the case $y = \frac{n-3}{2}$ and $z=1$ is not in the Apéry set $\mathrm{Ap}(S, \mathrm{F}(T))$. For this reason, in *5)*, there is one element less. This yields *4)* and *5)*. $\square$

Remark 8. *We left the proofs of the following curious facts to the reader.*

1) $M_i = \emptyset$ *for* $i \geq \frac{n+1}{2}$.

2) M_i *has* $6n - 12i - 1$ *elements for* $i \leq \frac{n-1}{2}$.

3) $\sum_{i \in \mathbb{N}} \# M_i = \mathrm{F}(T)$.

2.3. Restrictions on the construction of L*-shapes*

Let L be an L-shape associated with S. Conditions (C1) and (C2) imply that, if f is a factorization of $s \in S$ appearing in L, then any $f' \leq f$ corresponds to a factorization of an element $s' \in S$ (actually in the Apéry set of $\mathrm{F}(T)$). Moreover, f' is the only factorization of s' occurring in L.

With this idea in mind, we start showing that the minimal elements in $\mathsf{Z}(M_i)$ are enough to "control" all the elements in $\mathsf{Z}(M_i)$.

Lemma 9. *Let* $s \in \mathrm{Ap}(S, \mathrm{F}(T))$ *and let* $(\alpha, \beta, \gamma) \in \mathsf{Z}(s)$. *Then* $\mathrm{nf}(s) = (\alpha, \beta, \gamma)$ *if and only if* $\beta \leq \frac{n-1}{2}$ *and* $\gamma \in \{0, 1\}$.

Proof. This statement follows from lemmas 2 and 4. $\square$

Lemma 10. *Let i be a positive integer. If we take* $s, s' \in M_i$, *then the following facts are equivalent:*

1) $\mathrm{nf}(s') < \mathrm{nf}(s)$,

2) for any factorization $\zeta = (\alpha, \beta, \gamma) \in \mathsf{Z}(s)$ *there exists a unique* $\zeta' = (\alpha', \beta', \gamma') \in \mathsf{Z}(s')$ *such that* $\zeta' < \zeta$.

Proof. *2)* implies *1)* follows easily from the characterization of normal form given in Lemma 9.

Let us see now that statement *1)* implies *2)*. Write $\mathrm{nf}(s) = \mathrm{nf}(s') + (a, b, c)$, with $(a, b, c) \in \mathbb{N}^3 \setminus \{(0,0,0)\}$. It follows that $(a, b, c) + \mathsf{Z}(s') \subseteq \mathsf{Z}(s)$. As both sets have the same cardinality, i, we obtain an equality. Assertion *2)* now follows easily. $\square$

Let $\mathcal{M}_i = \{s \in M_i \mid \mathrm{nf}(s) \text{ is minimal in } \mathrm{nf}(M_i)\}$. From Lemma 10 and (C2), in order to construct an L-shape for every possible i, we only have to choose a factorization for each $s \in \mathcal{M}_i$.

The following result gives these minimal elements.

Lemma 11. $\mathrm{Minimals}_{\leq}(\mathrm{nf}(M_i)) = \left\{ \left(\frac{3n-5}{2}, i-2, 1\right), (3(i-1), i-1, 0) \right\}$.

Proof. From Lemma 7,

$$\mathrm{Minimals}_{\leq}(\mathrm{nf}(M_i)) = \mathrm{Minimals}_{\leq} \left\{ \left(\frac{3n-5}{2}, i-2, 1 \right), (3(i-1), i-1, 0), \right.$$
$$\left. (3(i-1), i-1, 1), (3(i-1), i, 0), (3(i-1), i, 1) \right\},$$

and the statement follows. $\square$

Corollary 12. *Each* $\mathcal{M}_i$ *has two elements.*

$$\mathcal{M}_i = \left\{ \mathbf{s}_i = \frac{3n-5}{2} n + (i-2)(3n-2) + 3n - 1,\ \mathbf{s}'_i = 3(i-1)n + (i-1)(3n-2) \right\}.$$

Next lemma gives an important reduction for the construction of L-shapes. By using this result, it is only necessary to choose a factorization involved in mp_1, that is, $(3,1,0)$ or $(0,0,2)$, to control all $\mathbf{s}'_i \in \mathcal{M}_i$, $i \in \{3, \ldots, \frac{n-1}{2}\}$.

Lemma 13. *Let L be an L-shape associated with $S = \langle n, 3n-2, 3n-1, F(T)\rangle$, and let $\mathbf{s} = 3n+(3n-2) = 2(3n-1)$ and $i \in \{3, \dots, \frac{n-1}{2}\}$.*

1) $\mathsf{Z}(\mathbf{s}) = \{(3,1,0),(0,0,2)\}$.

2) If $(3,1,0) \in L$, then $L \cap \mathsf{Z}(\mathbf{s}'_i) = \{(3(i-1), i-1, 0)\}$.

3) If $(0,0,2) \in L$, then $L \cap \mathsf{Z}(\mathbf{s}'_i) = \{(0,0,2(i-1))\}$.

In particular, $L \cap \mathsf{Z}(\mathbf{s}'_i) \subset \{(3(i-1), i-1, 0), (0,0,2(i-1))\}$.

Proof. The first assertion is already known, and it follows from (1) and the proof of [8, Theorem 10.25].

The factorizations of the element $\mathbf{s}'_\mathbf{i}$ are $(3x, x, 2(i-1-x))$ for $x \in \{0, \dots, i-1\}$. Take $x \in \{1, \dots, i-1\}$. If $(3x, x, 2(i-1-x)) \in L$, then $(3,1,0) \le (3x, x, 2(i-1-x))$ and $(0,0,2) \le (3x, x, 2(i-1-x))$. Hence Condition (C2) asserts that $(3,1,0),(0,0,2) \in L$, and thus $\#\mathsf{Z}(\mathbf{s}) \cap L = 2$, contradicting (C1). This means that none of these factorizations can be in L. Assertions *2)* and *3)* now follow easily by taking $x = 1$ and $x = i-1$. ☐

Notice that $\mathbf{s} = \mathbf{s}'_2$.

Lemma 14. *Let L be an L-shape associated with $S = \langle n, 3n-2, 3n-1, F(T)\rangle$, and let $i \in \{3, \dots, \frac{n-1}{2}\}$. Then $L \cap \mathsf{Z}(\mathbf{s}_i) \subset \left\{\left(\frac{3n-5}{2}, i-2, 1\right), \left(0, \frac{n-3}{2}+i, 0\right), \left(\frac{3n-5}{2} - 3(i-2), 0, 2(i-2)+1\right)\right\}$.*

Proof. The other factorizations of $\mathbf{s}_i$ are $\left(\frac{3n-5}{2} - 3x, i-2-x, 2x+1\right)$ with $x \in \{1, \dots, i-3\}$. Arguing as in Lemma 13, we deduce that they cannot be in L. ☐

Remark 15. *When $i = 2$, we have only two possible choices: $\left(0, \frac{n+1}{2}, 0\right)$ and $\left(\frac{3n-5}{2}, 0, 1\right)$. The first and third factorizations given by Lemma 14 are the same.*

2.4. A family of L-shapes

Now we will try to put, in an ordered way, the different possible factorizations for each element of $\{\mathbf{s}\} \cup \{\mathbf{s}_i \mid i \in \{2, \dots, \frac{n-1}{2}\}\}$. Such possible factorizations are given by Lemma 14.

For $i = \frac{n-1}{2}$, the element $\mathbf{s}_{\frac{n-1}{2}} = (n-2)(3n-2)$ has the following three factorizations to choose:

$$(0, n-2, 0), \left(\frac{3n-5}{2}, \frac{n-5}{2}, 1\right), (5, 0, n-4).$$

(a) The factorization $(0, n-2, 0)$ is over $\left(0, \frac{n-3}{2}+i, 0\right)$ for all possible i. So, when choosing $(0, n-2, 0)$, the L-shape L must contain the factorization $(0, \frac{n-3}{2}+i, 0)$ of the other possible $\mathbf{s}_i$'s. We can choose for $\mathbf{s}$ both of its factorizations, that is, either $(0,0,2)$ or $(3,1,0)$. Therefore, we can construct two different L-shapes from $(0, n-2, 0)$. The choices for $\mathbf{s}_i$ and $\mathbf{s}'_i$ are respectively (recall that the rest of factorizations of elements in $\mathrm{Ap}(S, \mathrm{F}(T))$ are forced by them):

$$\left(0, \frac{n+1}{2}, 0\right), \dots, \left(0, \frac{n-3}{2}+i, 0\right), \dots, (0, n-2, 0),$$
$$(0,0,2), \dots, (0,0,2(i-1)), \dots, (0,0,n-3),$$

or

$$\left(0, \frac{n+1}{2}, 0\right), \dots, \left(0, \frac{n-3}{2}+i, 0\right), \dots, (0, n-2, 0),$$
$$(3,1,0), \dots, (3(i-1), i-1, 0), \dots, \left(3\frac{n-3}{2}, \frac{n-3}{2}, 0\right).$$

(b) If we choose $\left(\frac{3n-5}{2}, \frac{n-5}{2}, 1\right)$ as the factorization of $\mathbf{s}_{\frac{n-2}{2}}$ in L, then all the other elements must be put in $\left(\frac{3n-5}{2}, i-2, 1\right)$. Even $\mathbf{s}$ must be put in $(3,1,0)$ because it is under $\left(\frac{3n-5}{2}, \frac{n-5}{2}, 1\right)$. So we only can construct one L-shape from $\left(\frac{3n-5}{2}, \frac{n-5}{2}, 1\right)$ (this construction corresponds to elements in F). The choices for $\mathbf{s}_i$ and $\mathbf{s}'_i$ are respectively

$$\left(\frac{3n-5}{2}, 0, 1\right), \ldots, \left(\frac{3n-5}{2}, i-2, 1\right), \ldots, \left(\frac{3n-5}{2}, \frac{n-5}{2}, 1\right),$$
$$(3,1,0), \ldots, (3(i-1), i-1, 0), \ldots, \left(3\frac{n-3}{2}, \frac{n-3}{2}, 0\right).$$

(c) Finally, if we choose $(5,0,n-4)$ for the last element $\mathbf{s}_{\frac{n-1}{2}}$, we can not take $(3,1,0)$ for $\mathbf{s}$, because $(0,0,2)$ is below $(5,0,n-4)$. So, as we cannot choose $(3,1,0)$ for $\mathbf{s}$, we also cannot select $\left(\frac{3n-5}{2}, i-2, 1\right)$ for $\mathbf{s_i}$, $i \in \{2, \ldots, \frac{n-1}{2}\}$, since $(3,1,0)$ is below these elements. Hence we can only choose for $\mathbf{s}_{\frac{n-3}{2}} = (n-3)(3n-2)$ two factorizations: $(0, n-3, 0)$ and $(8, 0, n-6)$. Both of them are feasible. Now, if we choose $(0, n-3, 0)$, the remaining elements are determined by this selection, as in (a). However, if we choose $(8,0,n-6)$, we have two new possibilities. Therefore, we obtain a new L-shape in each step. There are $\frac{n-5}{2}$ steps, so we have $\frac{n-3}{2}$ new L-shapes. The choices for $\mathbf{s}_i$ and $\mathbf{s}'_i$ are respectively:

$$\begin{array}{ll}
j=0, & \left(0, \frac{n+1}{2}, 0\right), \ldots, \left(0, \frac{n-3}{2}+i, 0\right), \ldots, (0, n-4, 0), (0, n-3, 0), (5, 0, n-4), \\
 & \qquad (0,0,2), \ldots, (0,0,2(i-1)), \ldots, (0,0,n-3); \\
j=1, & \left(0, \frac{n+1}{2}, 0\right), \ldots, \left(0, \frac{n-3}{2}+i, 0\right), \ldots, (0, n-4, 0), (8, 0, n-6), (5, 0, n-4), \\
 & \qquad (0,0,2), \ldots, (0,0,2(i-1)), \ldots, (0,0,n-3); \\
\vdots & \vdots \\
j=\frac{n-5}{2}, & \left(\frac{3n-5}{2}, 0, 1\right), \ldots, \left(\frac{3n-5}{2} - 3(i-2), 0, 2(i-2)+1\right), \ldots, (8,0,n-6), (5,0,n-4), \\
 & \qquad (0,0,2), \ldots, (0,0,2(i-1)), \ldots, (0,0,n-3).
\end{array}$$

Summarizing, we can obtain two L-shapes from $(0, n-2, 0)$, one more from $\left(\frac{3n-5}{2}, \frac{n-5}{2}, 1\right)$, and $\frac{n-3}{2}$ new L-shapes from $(5, 0, n-4)$. So, we can construct $2+1+\frac{n-3}{2} = \frac{n+3}{2}$ different L-shapes for $\langle n, 3n-2, 3n-1, \mathrm{F}(T)\rangle$.

Theorem 16. *The number of different* L*-shapes for an embedding dimension four numerical semigroup is not upper bounded.*

Proof. Consider $T_m = \langle m, 3m-2, 3m-1\rangle$ and $S_m = \langle m, 3m-2, 3m-1, \mathrm{F}(T_m)\rangle$ for odd $m \geq 5$. For any integer n greater than four, we can take $T_{2n-3} = \langle 2n-3, 3(2n-3)-2, 3(2n-3)-1\rangle$ and $S_{2n-3} = \langle 2n-3, 3(2n-3)-2, 3(2n-3)-1, \mathrm{F}(T_{2n-3})\rangle$. There exists, at least, $\frac{(2n-3)+3}{2} = n$ different L-shapes associated to S_{2n-3}. ☐

Figure 3 shows the L-shapes obtained for $m = 17$ and $n = 10$.

Remark 17. *From Lemma 7, we can obtain the maximal elements in each* M_i*. From these elements we can obtain the pseudo-Frobenius numbers of* S *because pseudo-Frobenius numbers are the maximal elements in the set* $\mathrm{Ap}(S_{2n-3}, \mathrm{F}(T_{2n-3})) - \mathrm{F}(T_{2n-3}) = \{w - \mathrm{F}(T_{2n-3}) \mid w \in \mathrm{Ap}(S_{2n-3}, \mathrm{F}(T_{2n-3}))\}$*, with respect to* $\leq_S$*. One can easily deduce that*

$$\mathrm{PF}(S) = \left\{\frac{3n^2-9n+4}{2}, \frac{3n^2-7n+2}{2}, \frac{3n^2-13n+8}{2}\right\}.$$

Remark 18. *Theorem 16 has his counterpart for weighted Cayley digraphs of degree three. More precisely, the digraph* $C(\mathrm{F}(T_{2n-3}); 2n-3, 6n-9, 6n-10; 2n-3, 6n-9, 6n-10)$*, for* $n \geq 4$*, have related* n *minimum distance diagrams.*

[1] F. Aguiló-Gost and P. A. García-Sánchez, Factoring in embedding dimension three numerical semigroups, *Electron. J. Comb.*, **17** (2010), no. 1, R#138, 21 pp.

[2] F. Aguiló-Gost and P. A. García-Sánchez, D. Llena An algorithm to compute the primitive elements of an embedding dimension three numerical semigroups, *Electron. Notes Disc. Math.*, **46** (2014), 185–192.

[3] H. Bresinsky, On prime ideals with generic zero $x_i = t^{n_i}$, *Proc. Amer. Math. Soc.* **47** (1975), 329–332.

[4] M. Delgado, P. A. García-Sánchez and J. Morais, "NumericalSgps", *A GAP package for numerical semigroups.* Available via (`http://www.gap-system.org`).

[5] M. A. Fiol, J. L. A. Yebra, I. Alegre and M. Valero, A discrete optimization problem in local networks and data alignment, *IEEE Trans. Comput.* **C-36** (1987) 702–713.

[6] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.7.5*; 2014, (`http://www.gap-system.org`).

[7] Ö. J. Rödseth. Weighted multi-connected loop networks, *Discrete Math.* **148** (1996) 161-173.

[8] J. C. Rosales and P. A. García-Sánchez. Numerical Semigroups, *Developments in Mathematics* 20. Springer, New York, 2009.

[9] P. Sabariego and F. Santos. Triple-loop networks with arbitrarily many minimum distance diagrams, *Discrete Math.* **309**(6) (2009) 1672–1684.

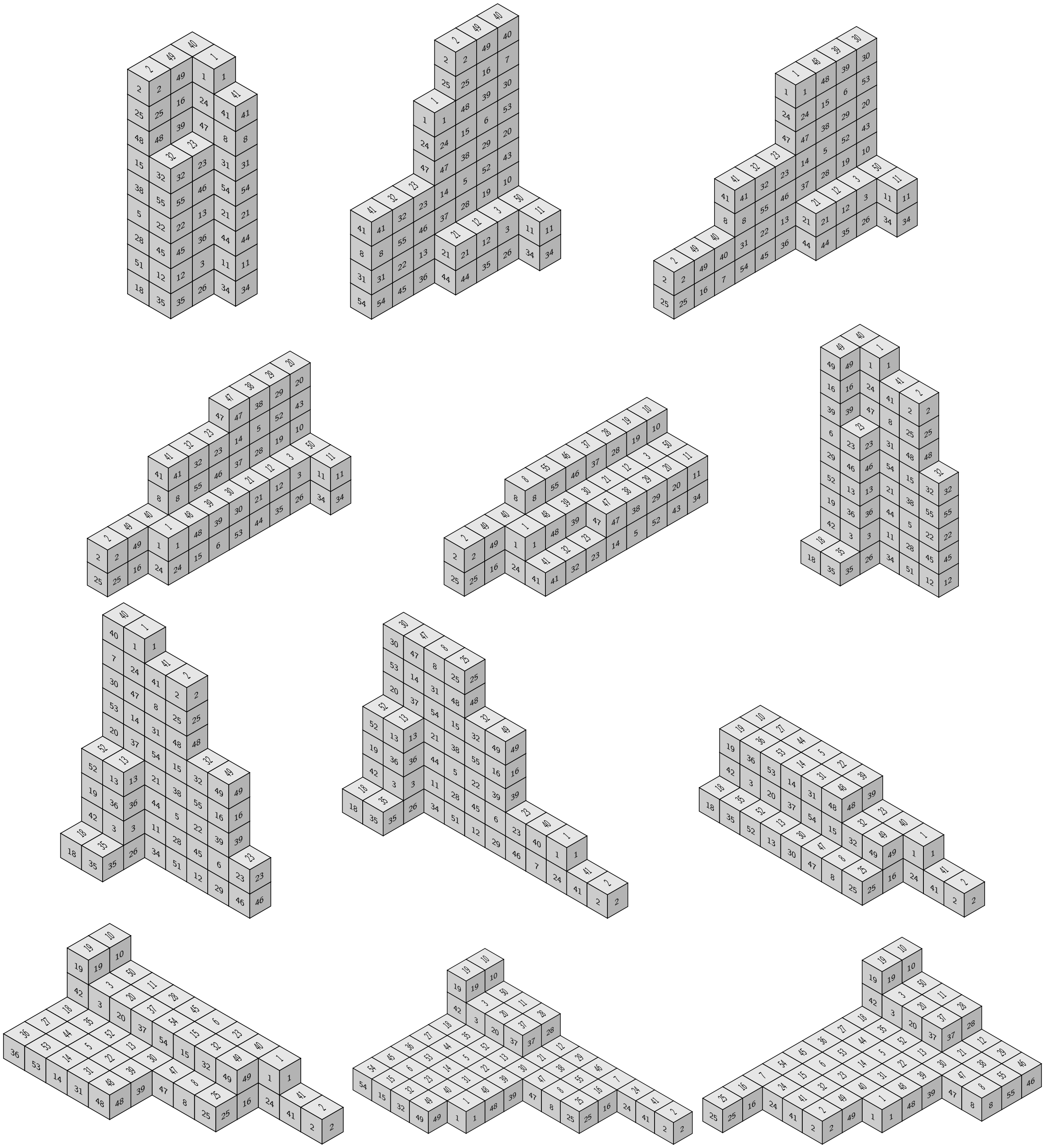


Figure 2: $C(\mathbb{Z}_{56}; 9, 17, 33; 1, 1, 1)$ and related MDDs

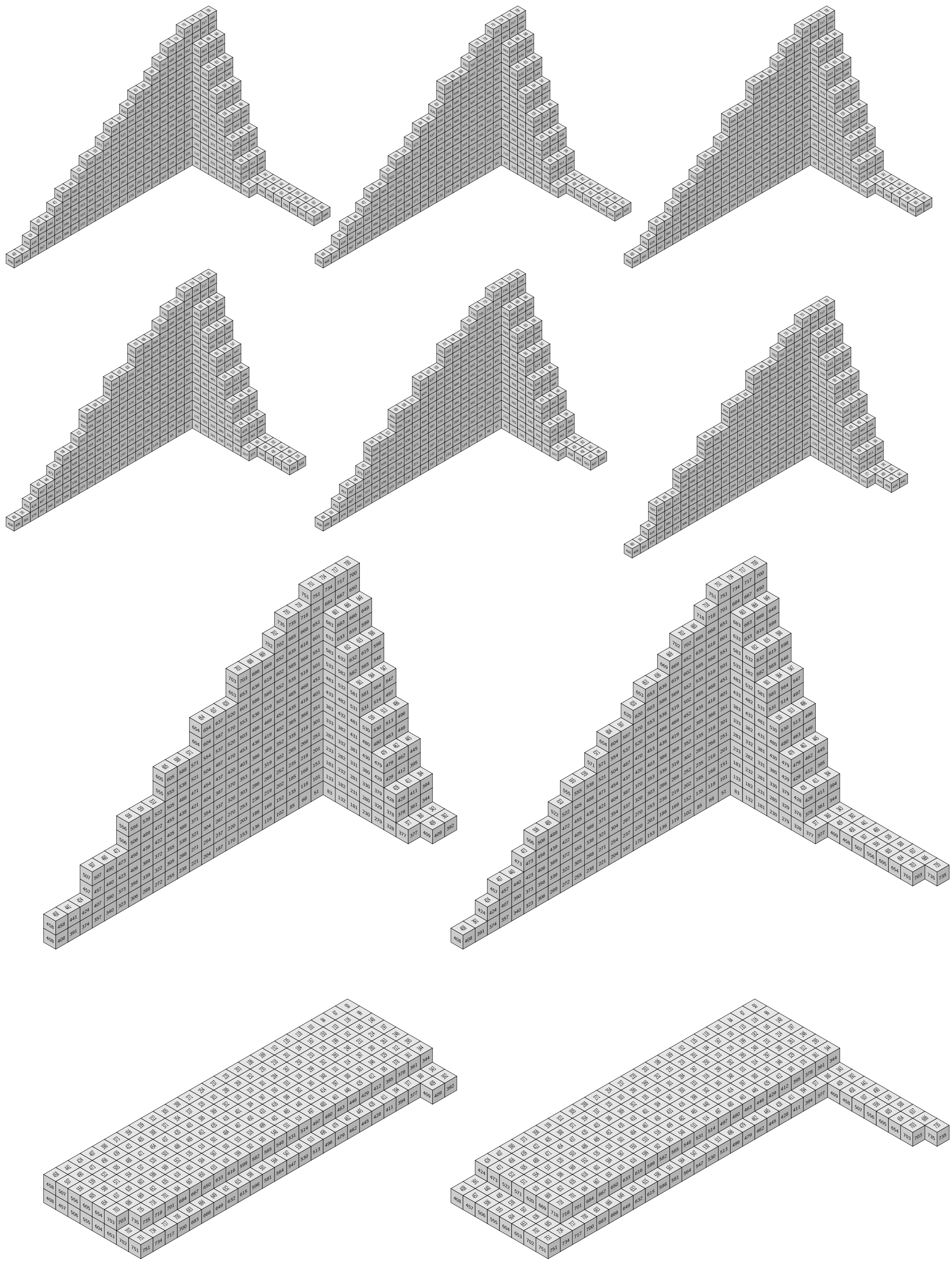


Figure 3: The set of L-shapes for $S = \langle 17, 49, 50, 376\rangle$